

# Classification of quantum superintegrable systems with quadratic integrals on two dimensional manifolds

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## Abstract

There are two classes of quantum integrable systems on a manifold with quadratic integrals, the Liouville and the Lie integrable systems as it happens in the classical case. The quantum Liouville quadratic integrable systems are defined on a Liouville manifold and the Schrödinger equation can be solved by separation of variables in one coordinate system. The Lie integrable systems are defined on a Lie manifold and are not generally separable ones but they can be solved. Therefore there are superintegrable systems with two quadratic integrals of motion not necessarily separable in two coordinate systems. The quantum analogues of the two dimensional superintegrable systems with quadratic integrals of motion on a manifold are classified by using the quadratic associative algebra of the integrals of motion. There are six general fundamental classes of quantum superintegrable systems corresponding to the classical ones. Analytic formulas for the involved integrals are calculated in all the cases. All the known quantum superintegrable systems are classified as special cases of these six general classes. The coefficients of the associative algebra of the general cases are calculated. These coefficients are the same as the coefficients of the classical case multiplied by  $-\hbar^2$  plus quantum corrections of order  $\hbar^4$  and  $\hbar^6$ .

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# I Introduction

In classical mechanics, a superintegrable or completely integrable is a Hamiltonian system with a maximum number of integrals. Two well known examples are the harmonic oscillator and the Coulomb potential. In the  $N$ -dimensional case the superintegrable system has  $2N - 1$  integrals, one among them is the Hamiltonian. In the two dimensional case the number of integrals in a superintegrable system is three.

The problem of complete classification of the superintegrable systems is recently achieved in the case where the integrals of motion are quadratic equations of the momenta [1, 2, 3]. All the known classical potentials can be classified in six classes [4]. This paper is essentially the quantum version of the reference [4].

In classical mechanics, if the potential is switched off, the problem is reduced to find the manifolds which possess more than one integrals. This problem was studied by Koenings in 19th century, see Darboux: *Leçons sur la Théorie Générale des Surfaces* [5]. In the same extremely detailed book Darboux classified the two dimensional manifolds with geodesics accepting two quadratic integrals (the Hamiltonian and one integral of motion) in two main classes, the Liouville and the Lie manifolds[5, vol III, n° 596, p. 38]. In modern language the systems defined on a manifold which are integrable with quadratic integrals of motion can be classified in two classes the Liouville and the Lie systems [2, 3, 4]. The Koenigs' main result was that, there are five classes of general forms of metrics, whose the geodesics have three integrals of motion (the Hamiltonian and two additional functionally independent integrals). These metrics are called "formes essentielles" and they depend on four parameters. All the metrics having more than two integrals of motion, can be obtained as partial cases of these "formes essentielles" by choosing appropriate values of the four parameters. The five classes of metrics are tabulated in "Tableau VII" by Koenigs[5, vol IV p.385].

The study of classical and quantum dimensional superintegrable systems with quadratic integrals of motion started with the investigation of superintegrable systems on the flat space and on spaces with constant curvature [6, 7]. In many instances, the notion of superintegrability was associated to the notion of separability on two different coordinate systems. All the separable classical and quantum systems are classified [1] and all the quadratic superintegrable systems in a flat space, on a space with constant curvature and on spaces by revolution are investigated [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. If the Hamiltonian has quadratic and non trivial linear terms then the superintegrability don't imply necessarily the separation of variables in two different coordinate systems [18]. This means that the notion of superintegrability

don't coincide with the notion of separability in two coordinate systems. In this paper the existence of non separable two dimensional integrable systems with quadratic integrals of motion is discussed.

The superintegrable systems corresponding to the Koenigs' "formes essentielles" were studied in [2, 3] where the integrals of motion are calculated. This "Tableau VII" of ref. [5] was created by studying the metrics possessing almost three integrals of motion on a manifold. In ref. [4] six classes of superintegrable systems are proved to exist. This result is corroborated by classifying the Staeckel equivalent superintegrable systems [19].

Each special case of a classical superintegrable system can be generated by fixing the associated constants in the appropriate class of Poisson algebra. This classification has been achieved by classifying all the possible Poisson quadratic algebras of the integrals. In this paper we show that, to each classical potential corresponds a quantum system and the Poisson algebra of integrals is replaced by a quadratic associative algebra as it was studied in [20, 21, 22, 8, 23]. The coefficients of the associative algebra are analytically calculated in all the cases as in ref. [4]. These coefficients are the similar as those, which were calculated in the classical case, but they are multiplied by  $-\hbar^2$ , the quantum coefficients are differentiated from the classical case by the presence of deformations of order  $\hbar^4$  and  $\hbar^6$ , which represent the quantum effects. The classification introduced in the classical case, which is based on the properties of the associative algebra, is an alternative method to the classifications based on the separation of variables [1] or to the study of the possible Darboux coefficients. These methods are very efficient if the manifold is fixed [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. The situation is more complicated in the case of a general manifold [2, 3, 4].

This paper is organized as follows: In section II the well known general forms of the quantum quadratic integrable systems on a manifold are derived for consistency reasons. The general solution of the Schrödinger equation for the Lie integrable systems is given. In Section III the quantum analogues of the classical superintegrable systems are discussed. To every classical system corresponds a quantum system and the quantum systems can be classified by classifying the corresponding quadratic associative algebra of integrals. This classification is similar to the classification of classic systems by using the quadratic Poisson algebra of integrals. In Section IV the potential and the integrals of the six classes of two dimensional quadratic superintegrable systems are calculated. The coefficients of the associative quadratic algebra of integrals is calculated and they are generated by the classical coefficients multiplied by  $-\hbar^2$  plus quantum corrections of order  $\hbar^4$  and  $\hbar^6$ . Finally in Section V the results of this paper and the open problems are shortly discussed.

## II Quantum integrable systems

Let us consider a two dimensional manifold with metric:

$$ds^2 = E(u, v)du^2 + 2F(u, v)dudv + G(u, v)dv^2 \quad (1)$$

In this coordinate system the quantum Hamiltonian has the general form

$$H = -\frac{\hbar^2}{2}\Delta + V \quad (2)$$

where  $\Delta$  is the Laplacian (or Laplace Beltrami operator)

$$\begin{aligned} \Delta = \text{div} \circ \text{grad} = & \frac{1}{\sqrt{EG - F^2}} \partial_u \left( \frac{G}{\sqrt{EG - F^2}} \partial_u - \frac{F}{\sqrt{EG - F^2}} \partial_v \right) + \\ & + \frac{1}{\sqrt{EG - F^2}} \partial_v \left( \frac{E}{\sqrt{EG - F^2}} \partial_v - \frac{F}{\sqrt{EG - F^2}} \partial_u \right) \end{aligned}$$

The above metric (1), using a conformal coordinate system, can be written as:

$$ds^2 = g(x, y) dx dy$$

In the conformal coordinate system the Hamiltonian (2) has the form

$$H = -\frac{\hbar^2}{2}\Delta + V = -\frac{\hbar^2}{g(x, y)} \partial_{xy} + V(x, y) \quad (3)$$

If the system possesses a quadratic integral of motion  $I$ , then under the above mentioned conformal coordinate system, the integral is an operator which has the form:

$$\begin{aligned} I = & -A(x, y) \hbar^2 \partial_{xx} - B(x, y) \hbar^2 \partial_{yy} + 2 \hbar^2 \frac{\beta(x, y)}{g(x, y)} \partial_{xy} \\ & -i \hbar r(x, y) \partial_x - i \hbar s(x, y) \partial_y + Q(x, y) \end{aligned}$$

By definition the Lie bracket between  $H$  and  $I$  must vanish:

$$[H, I] = HI - IH = 0 \quad (4)$$

The above equation implies restrictions on the involved functions of  $H$  and  $I$ . So, the coefficients of the partial derivatives  $\partial_{xxx}$  and  $\partial_{yyy}$  in equation (4) must be zero:

$$\begin{aligned} \frac{\partial A(x, y)}{\partial y} = 0 & \Rightarrow A = A(x) \\ \frac{\partial B(x, y)}{\partial x} = 0 & \Rightarrow B = B(y) \end{aligned} \quad (5)$$

Similarly, for the coefficients of  $\partial_{xx}$  and  $\partial_{yy}$  we have:

$$\begin{aligned}\frac{\partial r(x, y)}{\partial y} &= 0 \Rightarrow r = r(x) \\ \frac{\partial s(x, y)}{\partial x} &= 0 \Rightarrow s = s(y)\end{aligned}\tag{6}$$

The vanishing of the coefficients of  $\partial_{xxy}$  and  $\partial_{yyx}$  gives:

$$\frac{\partial \beta}{\partial y} = A(x) \frac{\partial g}{\partial x} + \frac{g}{2} A'(x)\tag{7}$$

$$\frac{\partial \beta}{\partial x} = B(y) \frac{\partial g}{\partial y} + \frac{g}{2} B'(y)\tag{8}$$

combining the last equations we get

$$g(x, y)(A''(x) - B''(y)) - 3B'(y) \frac{\partial g}{\partial y} - 2B(y) \frac{\partial^2 g}{\partial y^2} + 3A'(x) \frac{\partial g}{\partial x} + 2A(x) \frac{\partial^2 g}{\partial x^2} = 0\tag{9}$$

From the coefficient equation of  $\partial_{xy}$  taking into account the relations (7) and (8), we get

$$\begin{aligned}g(x, y) \left( -2i r'(x) - 2i s'(y) + \hbar(A''(x) + B''(y)) \right) + (-2i s(y) + \hbar B'(y)) \frac{\partial g}{\partial y} + \\ + (-2i r(x) + \hbar A'(x)) \frac{\partial g}{\partial x} = 0\end{aligned}$$

or

$$\frac{\partial}{\partial x} \left( (-2i r(x) + \hbar A'(x)) g(x, y) \right) + \frac{\partial}{\partial y} \left( (-2i s(y) + \hbar B'(y)) g(x, y) \right) = 0\tag{10}$$

and if we choose

$$r(x) = -\frac{1}{2}i \hbar A'(x), \quad s(y) = -\frac{1}{2}i \hbar B'(y)\tag{11}$$

the equation (10) is identically zero and the integral of motion  $I$  is written:

$$\begin{aligned}I = -\hbar^2 A(x) \partial_{xx} - \hbar^2 B(y) \partial_{yy} + 2\hbar^2 \frac{\beta(x, y)}{g(x, y)} \partial_{xy} - \frac{\hbar^2}{2} A'(x) \partial_x - \\ - \frac{\hbar^2}{2} B'(y) \partial_y + Q(x, y)\end{aligned}\tag{12}$$

The coefficients of  $\partial_x$  and  $\partial_y$  in (4) must be zero, so:

$$\frac{\partial Q}{\partial y} = 2g(x, y)A(x)\frac{\partial V}{\partial x} - 2\beta(x, y)\frac{\partial V}{\partial y} \quad (13)$$

$$\frac{\partial Q}{\partial x} = 2g(x, y)B(y)\frac{\partial V}{\partial x} - 2\beta(x, y)\frac{\partial V}{\partial y} \quad (14)$$

The partial derivative over  $x$  of equation (13) is equal to the derivative over  $y$  of equation (14), this equality imply:

$$\begin{aligned} g(x, y) \left( 3B'(y)\frac{\partial V}{\partial y} + 2B(y)\frac{\partial^2 V}{\partial y^2} - 3A'(x)\frac{\partial V}{\partial x} - 2A(x)\frac{\partial^2 V}{\partial x^2} \right) + \\ + 4B(y)\frac{\partial g}{\partial y}\frac{\partial V}{\partial y} - 4A(x)\frac{\partial g}{\partial x}\frac{\partial V}{\partial x} = 0 \end{aligned} \quad (15)$$

Equations (9) and (15) are the same as in the classical case [4], therefore we can distinguish two kinds of quantum superintegrable systems. The class I or Liouville integrable systems and the class II or Lie integrable systems.

**Class I: or Liouville systems:**  $A(x)B(y) \neq 0$

We choose a new coordinate system  $(\xi, \eta)$  with

$$\xi = \int \frac{dx}{\sqrt{A(x)}}, \quad \eta = \int \frac{dy}{\sqrt{B(y)}}$$

in this system the differential operators written as:

$$\partial_x = \frac{\partial \xi}{\partial x} \partial_\xi = \frac{1}{\sqrt{A(x)}} \partial_\xi, \quad \partial_y = \frac{\partial \eta}{\partial y} \partial_\eta = \frac{1}{\sqrt{B(y)}} \partial_\eta$$

therefore the Hamiltonian  $H$  and the integral  $I$  have the next form respectively:

$$\begin{aligned} H &= -\frac{\hbar^2}{g(\xi, \eta)} \partial_{\xi\eta} + V(\xi, \eta) \\ I &= -\hbar^2 \partial_{\xi\xi} - \hbar^2 \partial_{\eta\eta} + 2\hbar^2 \frac{\beta(\xi, \eta)}{g(\xi, \eta)} \partial_{\xi\eta} + Q(\xi, \eta) \end{aligned}$$

The coordinates  $(\xi, \eta)$  usually referred as *Liouville coordinates*, and the equation (9) is considerably simplified:

$$\frac{\partial^2 g}{\partial \xi^2} - \frac{\partial^2 g}{\partial \eta^2} = 0$$

with general solution

$$g(\xi, \eta) = F(\xi + \eta) + G(\xi - \eta)$$

Where  $F(u), G(v)$  arbitrary functions.

Now, from equations (7), (8) we can calculate the function  $\beta(\xi, \eta)$ . The above mentioned equations are simplified and they written as:

$$\frac{\partial \beta}{\partial \xi} = \frac{\partial g}{\partial \eta}, \quad \frac{\partial \beta}{\partial \eta} = \frac{\partial g}{\partial \xi}$$

therefore eliminating  $g(\xi, \eta)$  from the last equations we get

$$\frac{\partial^2 \beta}{\partial \xi^2} - \frac{\partial^2 \beta}{\partial \eta^2} = 0$$

therefore

$$\beta(\xi, \eta) = F(\xi + \eta) - G(\xi - \eta)$$

The equation (15) in Liouville coordinates is

$$\left( F(\xi + \eta) + G(\xi - \eta) \right) (V_{\xi\xi} - V_{\eta\eta}) + 2F'(\xi + \eta)(V_{\xi} - V_{\eta}) + 2G'(\xi - \eta)(V_{\xi} - V_{\eta}) = 0$$

where with  $V_x, V_y$  etc. denoted the partial derivatives of  $V(\xi, \eta)$ . The general solution of the last equation is

$$V(\xi, \eta) = \frac{f(\xi + \eta) + g(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)}$$

with  $f(u), g(v)$  arbitrary functions. Final the function  $Q(\xi, \eta)$  easily calculated from equations (13), (14):

$$Q(\xi, \eta) = 4 \frac{f(\xi + \eta)G(\xi - \eta) - g(\xi - \eta)F(\xi + \eta)}{F(\xi + \eta) + G(\xi - \eta)}$$

Introducing the corresponding isothermic coordinate system  $(u, v)$  which are defined by:

$$\xi = \frac{u + iv}{2}, \quad \eta = \frac{u - iv}{2}$$

The Hamiltonian  $H$  and the integral  $I$  can be written as:

$$H = \frac{1}{F(u) + G(v)} \left( (f(u) - \hbar^2 \partial_{uu}) + (g(v) - \hbar^2 \partial_{vv}) \right)$$

$$I = \frac{1}{F(u) + G(v)} (4G(v)(f(u) - \hbar^2 \partial_{uu}) - 4F(u)(g(v) - \hbar^2 \partial_{vv}))$$

The relations

$$H\Psi = E\Psi, \quad I\Psi = J\Psi$$

lead us to a formula for the Schrödinger function  $\Psi$ :

$$\frac{1}{F(u) + G(u)}(f(u) - \hbar^2 \partial_{uu})\Psi + \frac{1}{F(u) + G(u)}(g(v) - \hbar^2 \partial_{vv})\Psi = E\Psi$$

$$\frac{4G(v)}{F(u) + G(v)}(f(u) - \hbar^2 \partial_{uu})\Psi - \frac{4F(u)}{F(u) + G(v)}(g(v) - \hbar^2 \partial_{vv})\Psi = J\Psi$$

combining the last equations we get

$$4(f(u) - \hbar^2 \partial_{uu})\Psi = (4F(u)E + J)\Psi$$

$$4(g(v) - \hbar^2 \partial_{vv})\Psi = (4G(v)E - J)\Psi$$

and if we let

$$\Psi(u, v) = U(u)V(v)$$

the above equations separate their variables:

$$U''(u) = \frac{1}{4\hbar^2}(4f(u) - 4F(u)E - J)U(u)$$

$$V''(v) = \frac{1}{4\hbar^2}(4g(v) - 4G(v)E + J)V(v).$$

Therefore an integrable Class I (Liouville) quantum system can be solved by separation of variables.

**Class II or Lie systems:**  $B(y) = 0$

Defining a new coordinate system

$$\xi = \int \frac{dx}{\sqrt{A(x)}}, \quad \eta = y$$

the differential operators are

$$\partial_x = \frac{\partial \xi}{\partial x} \partial_\xi = \frac{1}{\sqrt{A(x)}} \partial_\xi, \quad \partial_y = \partial_\eta$$

In this case the Hamiltonian  $H$  and the integral  $I$  is written:

$$H = -\hbar^2 \frac{1}{g(\xi, \eta)} \partial_{\xi\eta} + V(\xi, \eta)$$

$$I = -\hbar^2 \partial_{\xi\xi} + 2\hbar^2 \frac{\beta(\xi, \eta)}{g(\xi, \eta)} \partial_{\xi\eta} + Q(\xi, \eta)$$



This specific coordinate system  $(\xi, \eta)$  is called *Lie coordinate system*. In Lie coordinates equation (9) is written:

$$\frac{\partial^2 g}{\partial \xi^2} = 0$$

The general solution of the last equation is

$$g(\xi, \eta) = F(\eta)\xi + G(\eta)$$

This metric defines a class of manifolds, which are called Lie manifolds. Similarly to the previous case the equations (7), (8) lead us to the next system:

$$\begin{aligned} \frac{\partial \beta}{\partial \eta} &= \frac{\partial g}{\partial \xi} \\ \frac{\partial \beta}{\partial \xi} &= 0 \end{aligned}$$

Taking into account the above expression for  $g(\xi, \eta)$ , we have

$$\beta(\xi, \eta) = \int_{\eta_0}^{\eta} F(\eta') d\eta'$$

Equation (15) is written:

$$\left( F(\eta)\xi + G(\eta) \right) \frac{\partial^2 V}{\partial \xi^2} + 2F(\eta) \frac{\partial V}{\partial \xi} = 0$$

with general solution given by the next expression

$$V(\xi, \eta) = \frac{f(\eta)\xi + g(\eta)}{F(\eta)\xi + G(\eta)}$$

where  $F(\eta)$ ,  $G(\eta)$ ,  $f(\eta)$ ,  $g(\eta)$  arbitrary functions. Equations (13), (14) imply:

$$Q(\xi, \eta) = \frac{-2(f(\eta)\xi + g(\eta))}{F(\eta)\xi + G(\eta)} \int_{\eta_0}^{\eta} F(\eta') d\eta' + 2 \int_{\eta_0}^{\eta} f(\eta') d\eta'$$

The relations

$$H\Psi = E\Psi, \quad I\Psi = J\Psi$$

lead us to the Schrödinger equation:

$$-\frac{\hbar^2 \Psi_{\xi\eta}}{F(\eta)\xi + G(\eta)} + \frac{f(\eta)\xi + g(\eta)}{F(\eta)\xi + G(\eta)} \Psi = E\Psi \quad (16)$$

The solution  $\Psi$  of this equation satisfies the following equation:

$$-\hbar^2 \Psi_{\xi\xi} + 2 \frac{\hbar^2 \int F(\eta) d\eta}{F(\eta)\xi + G(\eta)} \Psi_{\xi\eta} - 2 \frac{f(\eta)\xi + g(\eta)}{F(\eta)\xi + G(\eta)} \int F(\eta) d\eta \Psi + 2 \Psi \int f(\eta) d\eta = J \Psi$$

Taking into consideration Schrödinger equation (16), the above equation can be written:

$$-\hbar^2 \Psi_{\xi\xi} = \left( J + 2 \left( E \int_{\eta_0}^{\eta} F(\eta') d\eta' - \int_{\eta_0}^{\eta} f(\eta') d\eta' \right) \right) \Psi \quad (17)$$

Let put

$$\Pi(\eta) = J + 2 \left( E \int_{\eta_0}^{\eta} F(\eta') d\eta' - \int_{\eta_0}^{\eta} f(\eta') d\eta' \right), \quad p(\eta) = \sqrt{|\Pi(\eta)|}$$

We can find solutions comparable to the solutions given in WKB problems.

$$\text{If } \Pi(\eta) \geq 0 \rightsquigarrow \Psi(\eta) = A(\eta) e^{i \xi p(\eta)/\hbar} + B(\eta) e^{-i \xi p(\eta)/\hbar}$$

$$\begin{aligned} \text{where } A(\eta) &= \exp \left[ - \int_{\eta_0}^{\eta} \frac{-\hbar f(\eta') + E \hbar F(\eta') - i (-g(\eta') + E G(\eta')) p(\eta')}{\hbar \Pi(\eta')} d\eta' \right] \\ \text{and } B(\eta) &= \exp \left[ - \int_{\eta_0}^{\eta} \frac{-\hbar f(\eta') + E \hbar F(\eta') + i (-g(\eta') + E G(\eta')) p(\eta')}{\hbar \Pi(\eta')} d\eta' \right] \end{aligned}$$

$$\text{If } \Pi(\eta) < 0 \rightsquigarrow \Psi(\eta) = a(\eta) e^{\xi p(\eta)/\hbar} + b(\eta) e^{-\xi p(\eta)/\hbar}$$

$$\begin{aligned} \text{where } a(\eta) &= \exp \left[ - \int_{\eta_0}^{\eta} \frac{-\hbar f(\eta') + E \hbar F(\eta') - (-g(\eta') + E G(\eta')) p(\eta')}{\hbar \Pi(\eta')} d\eta' \right] \\ \text{and } b(\eta) &= \exp \left[ - \int_{\eta_0}^{\eta} \frac{-\hbar f(\eta') + E \hbar F(\eta') + (-g(\eta') + E G(\eta')) p(\eta')}{\hbar \Pi(\eta')} d\eta' \right] \end{aligned} \quad (18)$$

If we compare equations (9) and (15) with the corresponding ones for the classical two dimensional systems with quadratic integrals, they are indeed the same. Therefore we have shown the following Proposition:

**Proposition 1** *Any two dimensional classical integrable system on a manifold with quadratic integrals corresponds to a quantum integrable system.*

*There are two classes of quantum integrable systems, the Liouville systems defined on a Liouville manifold (Class I systems). These integrable systems can be solved by separation of variables. The other class of integrable systems are the Lie ones, which are defined on a Lie manifold (Class II systems). These integrable systems cannot be solved generally by separation of variables but they can be solved by using WKB like solutions.*

### III Quantum superintegrable systems

A system is superintegrable on a two dimensional manifold if it has three functionally independent integrals of motion  $H$ ,  $A$  and  $B$ . Let us consider a Hamiltonian  $H$  which in Liouville coordinate system has the form:

$$H = \frac{-\hbar^2}{g(\xi, \eta)} \partial_{\xi\eta} + V(\xi, \eta)$$

From the previous section we know that the integral of motion  $A$  of an integrable system, can be written in two specific forms analogous to two distinct coordinate systems which are:

$$A = -\hbar^2 \partial_{\xi\xi} - \hbar^2 \partial_{\eta\eta} + 2\hbar^2 \frac{\beta(\xi, \eta)}{g(\xi, \eta)} \partial_{\xi\eta} + Q(\xi, \eta) \quad (\text{Liouville system})$$

$$A = -\hbar^2 \partial_{\xi\xi} + 2\hbar^2 \frac{\beta(\xi, \eta)}{g(\xi, \eta)} \partial_{\xi\eta} + Q(\xi, \eta) \quad (\text{Lie system})$$

Let us consider the second integral of motion, in Liouville coordinates, of the form (12) namely,

$$B = -\hbar^2 A(\xi) \partial_{\xi\xi} - \hbar^2 B(\eta) \partial_{\eta\eta} + 2\hbar^2 \frac{\beta(\xi, \eta)}{g(\xi, \eta)} \partial_{\xi\eta} - \frac{\hbar^2}{2} A'(\xi) \partial_{\xi} - \frac{\hbar^2}{2} B'(\eta) \partial_{\eta} + Q(\xi, \eta)$$

therefore the following relations are satisfied:

$$[H, A] = [H, B] = 0$$

In this paper we study the quantum superintegrable systems where the integrals  $A$ ,  $B$  and  $C$  satisfy the quadratic associative algebra as in ref. [23]:

$$\begin{aligned} [A, B] &= C \\ [A, C] &= \alpha A^2 + \beta B^2 + \gamma \{A, B\} + \delta A + \epsilon B + \zeta \\ [B, C] &= a A^2 - \gamma B^2 - \alpha \{A, B\} + d A - \delta B + z \end{aligned} \quad (19)$$

where  $a, \gamma, \alpha, \beta$  are constants and

$$\begin{aligned}\delta &= \delta(H) = \delta_1 H + \delta_0, \\ \epsilon &= \epsilon_1 H + \epsilon_0, \\ \zeta &= \zeta(H) = \zeta_2 H^2 + \zeta_1 H + \zeta_0, \\ d &= d(H) = d_1 H + d_0, \\ z &= z(H) = z_2 H^2 + z_1 H + z_0,\end{aligned}$$

with  $\delta_i, \epsilon_i, \zeta_i, d_i$  and  $z_i$  constants. The Casimir of the above quadratic algebra is given by the expression:

$$\begin{aligned}K &= C^2 - \alpha\{A^2, B\} - \gamma\{A, B^2\} + \left(\alpha\gamma - \delta + \frac{a\beta}{3}\right)\{A, B\} - \\ &\quad - \frac{2\beta}{3}B^3 + \left(\gamma^2 - \epsilon - \frac{\alpha\beta}{3}\right)B^2 + \left(-\gamma\delta + 2\zeta - \frac{\beta d}{3}\right)B + \\ &\quad + \frac{2a}{3}A^3 + \left(d + \frac{a\gamma}{3} + \alpha^2\right)A^2 + \left(\frac{a\epsilon}{3} + \alpha\delta + 2z\right)A\end{aligned}\quad (20)$$

Since  $B$  is an integral of motion there exist a new coordinate system  $(X, Y)$ , let it be the Liouville one, in which the integral  $B$  has the following form:

$$B = -\hbar^2 \partial_{XX} - \hbar^2 \partial_{YY} + 2\hbar^2 \frac{\tilde{\beta}(X, Y)}{\tilde{g}(X, Y)} \partial_{XY} + \tilde{Q}(X, Y)$$

where  $\tilde{\beta}(X, Y), \tilde{Q}(X, Y)$  the functions  $\beta(\xi, \eta), Q(\xi, \eta)$  in  $(X, Y)$  coordinates and,

$$\tilde{g}(X, Y) = \frac{g(\xi, \eta)}{\sqrt{A(\xi)}\sqrt{B(\eta)}}$$

The integral  $B$  can be replaced by a linear combination of the integrals  $A, B$  and  $H$  and the coefficient  $\beta$  can be put always to be 0. In this case, the coefficients of  $(\partial_\xi)^6$  and  $(\partial_\eta)^6$  must vanish in the Casimir (20), so the following relations are true:

$$\begin{aligned}6\hbar^2(A'(\xi))^2 &= a - 3\gamma A^2(\xi) + 3\alpha A(\xi) \\ 6\hbar^2(B'(\eta))^2 &= a - 3\gamma B^2(\xi) + 3\alpha B(\xi)\end{aligned}\quad (21)$$

The superintegrable systems on a manifold can be classified by the solutions of the last equations. These equations differ from those of classical case only in a multiplier  $-\hbar^2$  as a coefficient of  $A'(\xi)$  and  $B'(\eta)$ . There are two classes of superintegrable systems. The class I systems corresponding to the case where both the integrals  $A$  and  $B$  are Liouville integrals and in the class II systems where the first integral  $A$  is of Lie type and the second is a

	$\alpha$	$\gamma$	$a$	$A(\xi)$	$B(\eta)$
$I_1$	0	0	$6\hbar^2$	$\xi$	$\eta$
$I_2$	$-8\hbar^2$	0	0	$\xi^2$	$\eta^2$
$I_3$	$32\hbar^2$	$-8\hbar^2$	0	$(e^\xi + e^{-\xi})^2$	$(e^\eta + e^{-\eta})^2$
$II_1$	0	0	0	1	0
$II_2$	0	0	$6\hbar^2$	$\xi$	0
$II_3$	$-8\hbar^2$	0	0	$\xi^2$	0

Table 1: Classification Table of quantum quadratic superintegrable systems ( $\beta = 0$ )

Liouville integral. Regarding equations (21) the superintegrable systems can be classified in the following six subclasses:

The associative algebra of integrals is characterized by the coefficients of the quantum superintegrable systems, these coefficients are exactly the same to the classical Poisson algebra constants multiplied to  $-\hbar^2$ .

Let us begin by the known solutions  $A(\xi)$ ,  $B(\eta)$  of equations (21). Taking into consideration that:

$$ds^2 = g(\xi, \eta) d\xi d\eta = (F(\xi + \eta) + G(\xi - \eta)) d\xi d\eta$$

Equation (9) is written for the Class I superintegrable systems

$$\begin{aligned} & (A''(\xi) - B''(\eta)) (F(\xi + \eta) + G(\xi - \eta)) + \\ & + 3A'(\xi) (F'(\xi + \eta) + G'(\xi - \eta)) - 3B'(\eta) (F'(\xi + \eta) - G'(\xi - \eta)) + \quad (22) \\ & + 2(A(\xi) - B(\eta)) (F''(\xi + \eta) + G''(\xi - \eta)) = 0 \end{aligned}$$

From the above equation the functions  $F(u)$  and  $G(v)$  are calculated by separation of variables for each subclass of quantum superintegrable system. We should notice that this equation is the same as in the classical case. Taking into consideration the general form of the potential in the case of Liouville integrable systems

$$V(\xi, \eta) = \frac{f(\xi + \eta) + g(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)}$$

equation (15) leads to a differential equation for the functions  $f(u)$  and  $g(v)$

which are involved in the definition of the potential:

$$\begin{aligned}
& f(\xi + \eta) \{-3 B'(\eta) (F'(\xi + \eta) - G'(\xi - \eta)) + 3 A'(\xi) (F'(\xi + \eta) + G'(\xi - \eta)) + \\
& + 2 (A(\xi) - B(\eta)) (F''(\xi + \eta) + G''(\xi - \eta))\} + \\
& + g(\xi - \eta) \{-3 B'(\eta) (F'(\xi + \eta) - G'(\xi - \eta)) + 3 A'(\xi) (F'(\xi + \eta) + G'(\xi - \eta)) + \\
& + 2 (A(\xi) - B(\eta)) (F''(\xi + \eta) + G''(\xi - \eta))\} - \\
& - (F(\xi + \eta) + G(\xi - \eta)) \{-3 B'(\eta) (f'(\xi + \eta) - g'(\xi - \eta)) + \\
& + 3 A'(\xi) (f'(\xi + \eta) + g'(\xi - \eta)) + 2 (A(\xi) - B(\eta)) (f''(\xi + \eta) + g''(\xi - \eta))\} = 0
\end{aligned} \tag{23}$$

We can eliminate from the above equation the functions  $F(u)$  and  $G(v)$ , which satisfy equation (22) and finally the functions involved in the definition of the potential satisfy the following equation:

$$\begin{aligned}
& (A''(\xi) - B''(\eta)) (f(\xi + \eta) + g(\xi - \eta)) + \\
& + 3A'(\xi) (g'(\xi + \eta) + g'(\xi - \eta)) - 3B'(\eta) (g'(\xi + \eta) - g'(\xi - \eta)) + \\
& + 2 (A(\xi) - B(\eta)) (f''(\xi + \eta) + g''(\xi - \eta)) = 0
\end{aligned} \tag{24}$$

This equation is indeed the same as (22). The general solutions of equations (22) and (24) are the same as in the classical case see ref [4]. These solutions are studied in Section IV.

For the class II integrable systems

$$ds^2 = (F(\eta)\xi + G(\eta)) d\xi d\eta$$

Equation (9) is written:

$$\begin{aligned}
& (A''(\xi) - B''(\eta)) (F(\eta)\xi + G(\eta)) + \\
& + 3A'(\xi)F(\eta) - 3B'(\eta) (F'(\eta)\xi + G'(\eta)) + 2 (A(\xi) - B(\eta)) (F''(\eta)\xi + G''(\eta)) = 0
\end{aligned} \tag{25}$$

From the above equation the functions  $F(\xi)$  and  $G(\eta)$  are calculated. This equation is the same as in the classical case. Taking into consideration the general form of the potential in the case of Lie integrable systems

$$V(\xi, \eta) = \frac{f(\eta)\xi + g(\eta)}{F(\eta)\xi + G(\eta)}$$

equation (15) leads to a complicated differential equation for the functions  $f(\xi)$  and  $g(\eta)$  analogous to equation (23) and after some algebra we find:

$$\begin{aligned}
& (A''(\xi) - B''(\eta)) (f(\eta)\xi + g(\eta)) + \\
& + 3A'(\xi)f(\eta) - 3B'(\eta) (f'(\eta)\xi + g'(\eta)) + 2 (A(\xi) - B(\eta)) (f''(\eta)\xi + g''(\eta)) = 0
\end{aligned} \tag{26}$$

The similarity of the pair of equations (22), (24) and (25), (26) to the corresponding ones in the classical case [4] leads to the following Proposition:

**Proposition 2** *Each classical superintegrable system with quadratic integrals corresponds to a quantum superintegrable system. There are six independent classes of quantum superintegrable integrals. The potentials, integrals and the coefficients of the associative algebra of the integrals (19) can be analytically calculated.*

## IV Classification of two dimensional superintegrable systems with two quadratic integrals of motion

In this section we give the analytical solutions for the different classes of superintegrable systems. As it was shown in Proposition 2, there are two general classes of superintegrable systems, each class has three subclasses. In this section the solutions of equations (22) and (24) and the coefficients of the associative integral algebra are calculated.

### IV-a Class I superintegrable systems

#### IV-a.1 Subclass I<sub>1</sub> of superintegrable systems

$$A(\xi) = \xi, \quad B(\eta) = \eta$$

$$\begin{aligned} F(u) &= 4\lambda u^2 + \kappa u + \nu/2, & G(v) &= -\lambda v^2 + \mu/v^2 + \nu/2 \\ f(u) &= 4\ell u^2 + k u + n/2, & g(v) &= -\ell v^2 + m/v^2 + n/2 \end{aligned} \quad (27)$$

$$ds^2 = g(\xi, \eta) d\xi d\eta, \quad g(\xi, \eta) = F(\xi + \eta) + G(\xi - \eta)$$

$$H = \frac{-\hbar^2}{g(\xi, \eta)} \partial_{\xi\eta} + V(\xi, \eta) \quad V(\xi, \eta) = \frac{w(\xi, \eta)}{g(\xi, \eta)}, \quad w(\xi, \eta) = f(\xi + \eta) + g(\xi - \eta)$$

The other integral of motion is:

$$\begin{aligned} A &= -\hbar^2 \partial_{\xi\xi} - \hbar^2 \partial_{\eta\eta} + 2\hbar^2 \frac{F(\xi + \eta) - G(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)} \partial_{\xi\eta} + \\ &+ 4 \frac{f(\xi + \eta) G(\xi - \eta) - g(\xi - \eta) F(\xi + \eta)}{F(\xi + \eta) + G(\xi - \eta)} \end{aligned}$$

We introduce the functions:

$$\begin{aligned}
\tilde{F}(u) &= \frac{\lambda u^6}{256} + \frac{\kappa u^4}{128} + \frac{\nu u^2}{16} - \frac{\mu}{u^2} \\
\tilde{G}(v) &= -\frac{\lambda v^6}{256} - \frac{\kappa v^4}{128} - \frac{\nu v^2}{16} + \frac{\mu}{v^2} \\
\tilde{f}(u) &= \frac{\ell u^6}{256} + \frac{k u^4}{128} + \frac{n u^2}{16} - \frac{m}{u^2} \\
\tilde{g}(v) &= -\frac{\ell v^6}{256} - \frac{k v^4}{128} - \frac{n v^2}{16} + \frac{m}{v^2}
\end{aligned} \tag{28}$$

The second integral of motion is:

$$\begin{aligned}
B = & -\hbar^2 \partial_{XX} - \hbar^2 \partial_{YY} + 2\hbar^2 \frac{\tilde{F}(X+Y) - \tilde{G}(X-Y)}{\tilde{F}(X+Y) + \tilde{G}(X-Y)} \partial_{XY} + \\
& + 4 \frac{\tilde{f}(X+Y)\tilde{G}(X-Y) - \tilde{g}(X-Y)\tilde{F}(X+Y)}{\tilde{F}(X+Y) + \tilde{G}(X-Y)}
\end{aligned}$$

where

$$X = 2\sqrt{\xi}, \quad \partial_X = \sqrt{\xi} \partial_\xi, \quad Y = 2\sqrt{\eta}, \quad \partial_Y = \sqrt{\eta} \partial_\eta$$

The constants of the Poisson algebra are:

$$\begin{aligned}
\alpha = 0 \quad \beta = 0 \quad \gamma = 0 \quad \delta = -16\hbar^2(\kappa H - k) \quad \epsilon = -256\hbar^2(\lambda H - \ell) \\
\zeta = 32\hbar^2(\kappa H - k)(\nu H - n) \quad a = 6\hbar^2 \quad d = -8(\nu H - n) \\
z = -8\hbar^2(\nu H - n)^2 + 128\hbar^2(\mu H - m)(\lambda H - \ell) - 96\hbar^4(\lambda H - \ell)
\end{aligned}$$

$$\begin{aligned}
K = & -32\hbar^2(\nu H - n)^3 - 512\hbar^2(\lambda H - \ell)(\nu H - n)(\mu H - m) + \\
& + 64\hbar^2(\kappa H - k)^2(\mu H - m) - 640\hbar^4(\lambda H - \ell)(\nu H - n) + 48\hbar^4(\kappa H - k)^2
\end{aligned}$$

#### IV-a.2 Subclass $I_2$ of superintegrable systems

$$A(\xi) = \xi^2, \quad B(\eta) = \eta^2$$

$$\begin{aligned}
F(u) &= \lambda u^2 + \frac{\kappa}{u^2} + \frac{\nu}{2}, \quad G(v) = -\lambda v^2 + \frac{\mu}{v^2} + \frac{\nu}{2} \\
f(u) &= \ell u^2 + \frac{k}{u^2} + \frac{n}{2}, \quad g(v) = -\ell v^2 + \frac{m}{v^2} + \frac{n}{2}
\end{aligned} \tag{29}$$



$$ds^2 = g(\xi, \eta) d\xi d\eta, \quad g(\xi, \eta) = F(\xi + \eta) + G(\xi - \eta)$$

$$H = \frac{-\hbar^2}{g(\xi, \eta)} \partial_{\xi\eta} + V(\xi, \eta) \quad V(\xi, \eta) = \frac{w(\xi, \eta)}{g(\xi, \eta)}, \quad w(\xi, \eta) = f(\xi + \eta) + g(\xi - \eta)$$

The other integral of motion is:

$$A = -\hbar^2 \partial_{\xi\xi} - \hbar^2 \partial_{\eta\eta} + 2\hbar^2 \frac{F(\xi + \eta) - G(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)} \partial_{\xi\eta} + 4 \frac{f(\xi + \eta)G(\xi - \eta) - g(\xi - \eta)F(\xi + \eta)}{F(\xi + \eta) + G(\xi - \eta)}$$

We introduce the functions:

$$\begin{aligned} \tilde{F}(u) &= 4\lambda e^{2u} + \nu e^u, \quad \tilde{G}(v) = \frac{\kappa e^v}{(1 + e^v)^2} + \frac{\mu e^v}{(-1 + e^v)^2} \\ \tilde{f}(u) &= 4\ell e^{2u} + n e^u, \quad \tilde{g}(v) = \frac{k e^v}{(1 + e^v)^2} + \frac{m e^v}{(-1 + e^v)^2} \end{aligned} \quad (30)$$

The second integral of motion is:

$$B = -\hbar^2 \partial_{XX} - \hbar^2 \partial_{YY} + 2\hbar^2 \frac{\tilde{F}(X + Y) - \tilde{G}(X - Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)} \partial_{XY} + 4 \frac{\tilde{f}(X + Y)\tilde{G}(X - Y) - \tilde{g}(X - Y)\tilde{F}(X + Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)}$$

where

$$X = \ln \xi, \quad \partial_X = \xi \partial_\xi, \quad Y = \ln \eta, \quad \partial_Y = \eta \partial_\eta$$

The constants of the Poisson algebra are:

$$\begin{aligned} \alpha &= -8\hbar^2 & \beta &= 0 & \gamma &= 0 & \delta &= 0 & \epsilon &= -256\hbar^2(\lambda H - \ell) \\ \zeta &= 32\hbar^2(\nu H - n)^2 - 256\hbar^2(\lambda H - \ell)((\mu H - m) - (\kappa H - k)) + 128\hbar^4(\lambda H - \ell) \\ a &= 0 & d &= 16\hbar^4 & z &= -32\hbar^2((\kappa H - k) + (\mu H - m))(\nu H - n) \end{aligned}$$

$$\begin{aligned} K &= -256\hbar^2(\lambda H - \ell)((\kappa H - k) + (\mu H - m))^2 - 128\hbar^2((\kappa H - k) - (\mu H - m)) \\ &(\nu H - n)^2 + 128\hbar^4((\nu H - n)^2 + 4(H\lambda - \ell)((\kappa H - k) - (\mu H - m))) + 4\hbar^6(\lambda H - \ell) \end{aligned}$$

#### IV-a.3 Subclass $I_3$ of superintegrable systems

$$A(\xi) = (e^\xi + e^{-\xi})^2, \quad B(\eta) = (e^\eta + e^{-\eta})^2$$

$$\begin{aligned}
F(u) &= \frac{\kappa e^{2u}}{(-1 + e^{2u})^2} + \frac{\lambda e^u (1 + e^{2u})}{(-1 + e^{2u})^2}, \quad G(v) = \frac{\mu e^{2v}}{(-1 + e^{2v})^2} + \frac{\nu e^v (1 + e^{2v})}{(-1 + e^{2v})^2} \\
f(u) &= \frac{k e^{2u}}{(-1 + e^{2u})^2} + \frac{\ell e^u (1 + e^{2u})}{(-1 + e^{2u})^2}, \quad g(v) = \frac{m e^{2v}}{(-1 + e^{2v})^2} + \frac{n e^v (1 + e^{2v})}{(-1 + e^{2v})^2} \\
ds^2 &= g(\xi, \eta) d\xi d\eta, \quad g(\xi, \eta) = F(\xi + \eta) + G(\xi - \eta)
\end{aligned} \tag{31}$$

$$H = \frac{-\hbar^2}{g(\xi, \eta)} \partial_{\xi\eta} + V(\xi, \eta) \quad V(\xi, \eta) = \frac{w(\xi, \eta)}{g(\xi, \eta)}, \quad w(\xi, \eta) = f(\xi + \eta) + g(\xi - \eta)$$

The other integral of motion is:

$$\begin{aligned}
A &= -\hbar^2 \partial_{\xi\xi} - \hbar^2 \partial_{\eta\eta} + 2\hbar^2 \frac{F(\xi + \eta) - G(\xi - \eta)}{F(\xi + \eta) + G(\xi - \eta)} \partial_{\xi\eta} + \\
&\quad + 4 \frac{f(\xi + \eta) G(\xi - \eta) - g(\xi - \eta) F(\xi + \eta)}{F(\xi + \eta) + G(\xi - \eta)}
\end{aligned}$$

We introduce the functions:

$$\begin{aligned}
\tilde{F}(u) &= \frac{(\kappa + 2\lambda)}{4} \tan^2 u + \frac{2\nu - \mu}{4} \cot^2 u + \frac{\lambda + \nu}{2} \\
\tilde{G}(v) &= \frac{(2\lambda - \kappa)}{4} \tan^2 v + \frac{\mu + 2\nu}{4} \cot^2 v + \frac{\lambda + \nu}{2} \\
\tilde{f}(u) &= \frac{(k + 2\ell)}{4} \tan^2 u + \frac{2n - m}{4} \cot^2 u + \frac{\ell + n}{2} \\
\tilde{g}(v) &= \frac{(2\ell - k)}{4} \tan^2 v + \frac{m + 2n}{4} \cot^2 v + \frac{\ell + n}{2}
\end{aligned} \tag{32}$$

The second integral of motion is:

$$\begin{aligned}
B &= -\hbar^2 \partial_{XX} - \hbar^2 \partial_{YY} + 2\hbar^2 \frac{\tilde{F}(X + Y) - \tilde{G}(X - Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)} \partial_{XY} + \\
&\quad + 4 \frac{\tilde{f}(X + Y) \tilde{G}(X - Y) - \tilde{g}(X - Y) \tilde{F}(X + Y)}{\tilde{F}(X + Y) + \tilde{G}(X - Y)}
\end{aligned}$$

where

$$X = \arctan(e^\xi), \quad \partial_X = (e^\xi + e^{-\xi}) \partial_\xi, \quad Y = \arctan(e^\eta), \quad \partial_Y = (e^\eta + e^{-\eta}) \partial_\eta$$

The constants of the Poisson algebra are:

$$\begin{aligned}
\alpha &= 32\hbar^2 & \beta &= 0 & \gamma &= -8\hbar^2 & \delta &= 32\hbar^4 & \epsilon &= -16\hbar^4 & \zeta &= 32\hbar^2 (\lambda H - \ell) (\nu H - n) \\
a &= 0 & d &= -64\hbar^2 (\kappa H - k) + 64\hbar^2 (\mu H - \mu) + 256\hbar^4 \\
z &= -32\hbar^2 ((\lambda - \nu)H - (\ell - n))^2 + 32\hbar^2 ((\kappa H - k) (\mu H - m)) + 32\hbar^4 (\mu H - m) - \\
&\quad - 32\hbar^4 (\kappa H - k) \\
K &= -64\hbar^2 (\kappa H - k) (\nu H - n)^2 + 64\hbar^2 (\lambda H - \ell)^2 (\mu H - m) - 512\hbar^4 (\nu H - n) \\
&\quad (\lambda H - \ell) - 64\hbar^4 (\mu H - m) (\kappa H - k) + 128\hbar^4 (\lambda H - \ell)^2 + 128\hbar^4 (\nu H - n) + \\
&\quad + 128\hbar^6 (\kappa H - k) - 128\hbar^6 (\mu H - m)
\end{aligned}$$

## IV-b Class II superintegrable systems

### IV-b.1 Subclass II<sub>1</sub> of superintegrable systems

$$A(\xi) = 1, \quad B(\eta) = 1$$

$$\begin{aligned}
F(\eta) &= \kappa \eta + \lambda, & G(\eta) &= \mu \eta + \nu \\
f(\eta) &= k \eta + \ell, & g(\eta) &= m \eta + n
\end{aligned} \tag{33}$$

$$ds^2 = g(\xi, \eta) d\xi d\eta, \quad g(\xi, \eta) = \xi F(\eta) + G(\eta)$$

$$H = \frac{-\hbar^2}{g(\xi, \eta)} \partial_{\xi\eta} + V(\xi, \eta) \quad V(\xi, \eta) = \frac{w(\xi, \eta)}{g(\xi, \eta)}, \quad w(\xi, \eta) = \xi f(\eta) + g(\eta)$$

The other integral of motion is:

$$A = -\hbar^2 \partial_{\xi\xi} + \frac{2\hbar^2 \int_{\eta_0}^{\eta} F(\eta') d\eta'}{g(\xi, \eta)} \partial_{\xi\eta} - \frac{2(\xi f(\eta) + g(\eta)) \int_{\eta_0}^{\eta} F(\eta') d\eta'}{g(\xi, \eta)} + 2 \int_{\eta_0}^{\eta} f(\eta') d\eta'$$

We introduce the functions:

$$\begin{aligned}
\tilde{F}(u) &= \frac{\kappa u^2}{4} + \frac{(\lambda + \mu) u}{2} + \frac{\nu}{2} \\
\tilde{G}(v) &= -\frac{\kappa v^2}{4} + \frac{(\lambda - \mu) v}{2} + \frac{\nu}{2} \\
\tilde{f}(u) &= \frac{k u^2}{4} + \frac{(\ell + m) u}{2} + \frac{n}{2} \\
\tilde{g}(v) &= -\frac{k v^2}{4} + \frac{(\ell - m) v}{2} + \frac{n}{2}
\end{aligned} \tag{34}$$

The second integral of motion is:

$$B = -\hbar^2 \partial_{\xi\xi} - \hbar^2 \partial_{\eta\eta} + 2\hbar^2 \frac{\tilde{F}(\xi + \eta) - \tilde{G}(\xi - \eta)}{\tilde{F}(\xi + \eta) + \tilde{G}(\xi - \eta)} \partial_{\xi\eta} +$$

$$+ 4 \frac{\tilde{f}(\xi + \eta) \tilde{G}(\xi - \eta) - \tilde{g}(\xi - \eta) \tilde{F}(\xi + \eta)}{\tilde{F}(\xi + \eta) + \tilde{G}(\xi - \eta)}$$

The constants of the Poisson algebra are:

$$\alpha = 0 \quad \beta = 0 \quad \gamma = 0 \quad \delta = -8\hbar^2(k - \kappa H) \quad \epsilon = 0 \quad \zeta = -8\hbar^2(\lambda H - \ell)^2$$

$$a = 0 \quad d = -16\hbar^2(k - \kappa H) \quad z = -8\hbar^2(\lambda H - \ell)^2 + 8\hbar^2(\mu H - m)^2$$

$$K = -16\hbar^2(\nu H - n)^2(\kappa H - k) + 32\hbar^2(\lambda H - \ell)(\mu H - m)(\nu H - n) - 16\hbar^4(k - \kappa H)^2$$

#### IV-b.2 Subclass II<sub>2</sub> of superintegrable systems

$$A(\xi) = \xi, \quad B(\eta) = \eta$$

$$F(\eta) = \frac{\kappa}{\sqrt{\eta}} + \lambda, \quad G(\eta) = 3\kappa\sqrt{\eta} + \lambda\eta + \frac{\mu}{\sqrt{\eta}} + \nu \quad (35)$$

$$f(\eta) = \frac{k}{\sqrt{\eta}} + \ell, \quad g(\eta) = 3k\sqrt{\eta} + \ell\eta + \frac{m}{\sqrt{\eta}} + n$$

$$ds^2 = g(\xi, \eta) d\xi d\eta, \quad g(\xi, \eta) = \xi F(\eta) + G(\eta)$$

$$H = \frac{-\hbar^2}{g(\xi, \eta)} \partial_{\xi\eta} + V(\xi, \eta) \quad V(\xi, \eta) = \frac{w(\xi, \eta)}{g(\xi, \eta)}, \quad w(\xi, \eta) = \xi f(\eta) + g(\eta)$$

The other integral of motion is:

$$A = -\hbar^2 \partial_{\xi\xi} + \frac{2\hbar^2 \int_{\eta_0}^{\eta} F(\eta') d\eta'}{g(\xi, \eta)} \partial_{\xi\eta} - \frac{2(\xi f(\eta) + g(\eta)) \int_{\eta_0}^{\eta} F(\eta') d\eta'}{g(\xi, \eta)} + 2 \int_{\eta_0}^{\eta} f(\eta') d\eta'$$

We introduce the functions:

$$\begin{aligned}
\tilde{F}(u) &= \frac{\lambda u^4}{128} + \frac{\kappa u^3}{16} + \frac{\nu u^2}{16} + \frac{\mu u}{4} \\
\tilde{G}(v) &= -\frac{\lambda v^4}{128} + \frac{\kappa v^3}{16} + \frac{\mu v}{4} - \frac{\nu v^2}{16} \\
\tilde{f}(u) &= \frac{\ell u^4}{128} + \frac{k u^3}{16} + \frac{n u^2}{16} + \frac{m u}{4} \\
\tilde{g}(v) &= -\frac{\ell v^4}{128} + \frac{k v^3}{16} + \frac{m v}{4} - \frac{n v^2}{16}
\end{aligned} \tag{36}$$

The second integral of motion is:

$$\begin{aligned}
B = & -\hbar^2 \partial_{XX} - \hbar^2 \partial_{YY} + 2\hbar^2 \frac{\tilde{F}(X+Y) - \tilde{G}(X-Y)}{\tilde{F}(X+Y) + \tilde{G}(X-Y)} \partial_{XY} + \\
& + 4 \frac{\tilde{f}(X+Y)\tilde{G}(X-Y) - \tilde{g}(X-Y)\tilde{F}(X+Y)}{\tilde{F}(X+Y) + \tilde{G}(X-Y)}
\end{aligned}$$

where

$$X = 2\sqrt{\xi}, \quad \partial_X = \sqrt{\xi} \partial_\xi, \quad Y = 2\sqrt{\eta}, \quad \partial_Y = \sqrt{\eta} \partial_\eta$$

The constants of the Poisson algebra are:

$$\begin{aligned}
\alpha = 0 \quad \beta = 0 \quad \gamma = 0 \quad \delta = -4\hbar^2 (\ell - \lambda H) \quad \epsilon = 0 \quad \zeta = -8\hbar^2 (\kappa H - k)^2 \\
a = 6\hbar^2 \quad d = -8\hbar^2 (\nu H - n) \quad z = 8\hbar^2 (\kappa H - k) (\mu H - m) + 2\hbar^2 (\nu H - n)^2 \\
K = -8\hbar^2 (\lambda H - \ell) (\mu H - m)^2 + 16\hbar^2 (\kappa H - k) (\mu H - m) (\nu H - n) - 4\hbar^4 (\ell - H\lambda)^2
\end{aligned}$$

#### IV-b.3 Subclass II<sub>3</sub> of superintegrable systems

$$A(\xi) = \xi^2, \quad B(\eta) = \eta^2$$

$$F(\eta) = \lambda \eta + \frac{\kappa}{\eta^3}, \quad G(\eta) = \nu + \frac{\mu}{\eta^2} \tag{37}$$

$$f(\eta) = \ell \eta + \frac{k}{\eta^3}, \quad g(\eta) = n + \frac{m}{\eta^2}$$

$$ds^2 = g(\xi, \eta) d\xi d\eta, \quad g(\xi, \eta) = \xi F(\eta) + G(\eta)$$

$$H = \frac{-\hbar^2}{g(\xi, \eta)} \partial_{\xi\eta} + V(\xi, \eta) \quad V(\xi, \eta) = \frac{w(\xi, \eta)}{g(\xi, \eta)}, \quad w(\xi, \eta) = \xi f(\eta) + g(\eta)$$

The other integral of motion is:

$$A = -\hbar^2 \partial_{\xi\xi} + \frac{2\hbar^2 \int_{\eta_0}^{\eta} F(\eta') d\eta'}{g(\xi, \eta)} \partial_{\xi\eta} - \frac{2(\xi f(\eta) + g(\eta)) \int_{\eta_0}^{\eta} F(\eta') d\eta'}{g(\xi, \eta)} + 2 \int_{\eta_0}^{\eta} f(\eta') d\eta'$$

We introduce the functions:

$$\begin{aligned} \tilde{F}(u) &= \lambda e^{2u} + \nu e^u, & \tilde{G}(v) &= \kappa e^{2v} + \mu e^v \\ \tilde{f}(u) &= \ell e^{2u} + n e^u, & \tilde{g}(v) &= k e^{2v} + m e^v \end{aligned} \quad (38)$$

The second integral of motion is:

$$\begin{aligned} B = & -\hbar^2 \partial_{XX} - \hbar^2 \partial_{YY} + 2\hbar^2 \frac{\tilde{F}(X+Y) - \tilde{G}(X-Y)}{\tilde{F}(X+Y) + \tilde{G}(X-Y)} \partial_{XY} + \\ & + 4 \frac{\tilde{f}(X+Y)\tilde{G}(X-Y) - \tilde{g}(X-Y)\tilde{F}(X+Y)}{\tilde{F}(X+Y) + \tilde{G}(X-Y)} \end{aligned}$$

where

$$X = \ln \xi, \quad \partial_X = \xi \partial_{\xi}, \quad Y = \ln \eta, \quad \partial_Y = \eta \partial_{\eta}$$

The constants of the Poisson algebra are:

$$\begin{aligned} \alpha &= -8\hbar^2, & \beta &= 0, & \gamma &= 0, & \delta &= 0, & \epsilon &= 0, & \zeta &= -32\hbar^2 (\kappa H - k) (\lambda H - \ell), \\ a &= 0, & d &= 16\hbar^4, & z &= -32\hbar^2 (\mu H - m) (\nu H - n) \end{aligned}$$

$$K = -64\hbar^2 (\lambda H - \ell) (\mu H - m)^2 + 64\hbar^2 (\kappa H - k) (\nu H - n)^2 + 64\hbar^4 (k - H\kappa)(H\lambda - \ell)$$

## V Discussion

In this paper the quantum superintegrable system with quadratic integrals of motion are classified in six classes as the classical ones. The potentials, integrals and the coefficients of the associative algebra of the integrals (19) are calculated. The problem of finding and classify of the quadratic superintegrable systems is related to the algebraic problem of classifying quadratic Poisson algebras in the classical case. In the quantum case the notion of superintegrability is ambiguous, therefore the quantum analogues of the classical systems are constructed and the quantum analogues of the Poisson quadratic algebra is the associative quadratic algebra of integrals. The superintegrability don't imply necessarily separation of variables in two different systems because the superintegrable systems possessing a Liouville and a Lie integral

are not necessarily separable in two systems, but they are solvable as it was proved in Section II. The existence of nonseparable superintegrable systems has been shown in the case of superintegrable systems with Hamiltonian having nontrivial linear in momenta terms plus the quadratic ones [18].

There are several open problems arising from the study of the two dimensional superintegrable systems. We shall indicate some of these problems

1. The calculation of the energy eigenvalues of these systems is related to the finite dimensional representations of the quadratic algebra of integrals [23]. Therefore the mathematical problem of the study of representation theory of quadratic and polynomial extensions of the Lie algebras is an open mathematical problem. The representations of the unitary representations of the quadratic algebra related to the four classical superintegrable systems have been studied in [23], using the techniques of lowering and uppering operators which is related to the deformed oscillator algebra [24]. The importance of quadratic or cubic associative algebras in the integrable and solvable problems and the discussion of a representation theory can be found in [25, 26].
2. The classification of two dimensional systems with a cubic and a quadratic integral is an open problem. The cubic problems have been investigated recently [27, 28, 2, 3, 31, 32, 29, 30].
3. Recently the problem of the classification of the classical and quantum three dimensional in conformally flat spaces has been investigated [31] and [32]. The general theory of the superintegrability in a general manifold is one of the most interesting open problems.

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